not a consequence of *assuming* (6.2), but an unavoidable fact, since a 4-vector relation [namely (6.2)] if valid in S must be valid also in S′.

For further reassurance, consider two identical particles moving along the respective z-axes of the usual two inertial frames S and S′ in opposite directions but with equal speeds u. Then, by (3.6)(iii), each has a z-velocity numerically equal to $u\gamma^{-1}(v)$ as judged from the other frame. Let the particles collide and fuse at the momentarily coincident origins of S and S′. By symmetry, the new compound particle can have no z-velocity in either frame. Thus conservation of z-momentum in S in accordance with a formula of type (6.6) requires

$$m(u)u = \frac{m(w)u}{\gamma(v)},$$

where $m(u)$ denotes the mass of one of the original particles at speed u, and w is the total speed in S of the particle moving on the z'-axis. Canceling u and then letting $u \to 0$ (that is, considering a sequence of experiments with ever smaller u) forces us to conclude

$$m(0) = \frac{m(v)}{\gamma(v)}.$$

This very directly shows that if we wish to salvage the *form* (6.6) of Newton's law of momentum conservation, we must allow m to vary precisely as in (6.4). And that, in a nutshell, is the crucial difference between relativistic and pre-relativistic collision mechanics.

6.3 The equivalence of mass and energy

Let us now take a closer look at eqn (6.7), $\sum^* m = 0$, the conservation of relativistic mass. At first sight this appears to be just an analog of the Newtonian law of mass conservation—but it is not! Newton defined mass as 'quantity of matter', and asserting its conservation was tantamount to asserting that matter can neither be created nor destroyed. But we now know that matter *can* be transmuted into radiation, as when an electron and a positron annihilate each other. So it is just as well that $\sum^* m = 0$ is not, in fact, an analog of the Newtonian law, except in a purely formal sense. What is conserved here is a quantity, γm_0, which varies with speed. In classical mechanics we know of only one such conserved quantity, namely the kinetic energy of particles in an elastic collision. Of course, (6.7) must hold in *all* collisions, elastic or not, if our approach is right; and we already know for a fact that it holds in all those fast collisions that are slow in *some* inertial frame. Could it be that m (or a multiple of it) is a measure of *total* energy? The answer turns out to be 'yes' and was regarded by Einstein, who found it in 1905 (but see the last paragraph of this section), as the most significant result of his special theory of relativity. Nevertheless, Einstein's assertion of the full equivalence of mass and energy, according to the famous formula

$$E = mc^2, \tag{6.8}$$

was in part a hypothesis, as we shall see. It cannot be uniquely deduced from the other axioms.

Consider the following expansion for the mass (6.4):

$$m = m_0\left(1 - \frac{u^2}{c^2}\right)^{-1/2} = m_0 + \frac{1}{c^2}\left(\frac{1}{2}m_0u^2\right) + \cdots \tag{6.9}$$

This shows that the relativistic mass of a slowly moving particle exceeds its rest-mass by $1/c^2$ times its kinetic energy (assuming the approximate validity of the Newtonian expression for the latter). So kinetic energy *contributes* to the mass in a way that is consistent with (6.8). In fact, it is eqn (6.9) that supplies the constant of proportionality between E and m. And it is the enormity of this constant that explains why the mass-increases correponding to the easily measurable kinetic energies of particles in classical collisions had never been observed.

We can next show that, since kinetic energy 'has mass', all energy must have mass in the same proportion. For one of the characteristics of energy is its transmutability from one form to another. When two oppositely moving identical particles collide and fuse and remain at rest (we are here thinking of putty balls rather than protons) $\sum m$ remains constant throughout, so that whatever mass was contributed before impact by the kinetic energy is thereafter contributed by the equal amount of thermal energy into which it changes. But then *all* forms of energy must have mass in the same proportion. For inside the now stationary compound particle the extra heat can be transmuted arbitrarily into other forms of energy without setting the particle in motion; for each such transmutation-event can be regarded as a 'collision', in which the total momentum is conserved; but also the total mass is conserved, which proves our assertion.

Yet it is still logically possible that energy only *contributes* to mass, without causing *all* of it. Especially in Einstein's time it would have been perfectly reasonable to suppose that the elementary particles are indestructible, so that the *available* energy of a macroscopic particle would be $c^2(m - q)$, where q is the total rest-mass of its constituent elementary particles. To equate *all* mass with energy required an act of aesthetic faith, very characteristic of Einstein. Of course, today we know how amply nature has confirmed that faith.

Einstein's mass–energy equivalence is not restricted to mechanics. It has been found applicable in many other branches of physics, from electromagnetism to general relativity. It is truly a new fundamental principle of Nature.

Observe also how this principle determines a *zero-point* of energy. In Newton's theory, for example, one could theoretically extract an unlimited amount of energy from a macroscopic body by letting it collapse indefinitely under its own gravity. According to Einstein, on the other hand, Nature must find a mechanism to prevent the extraction of more than mc^2 units of energy. That mechanism is the general-relativistic 'black hole'!

The relativistic *kinetic energy* T of a particle is naturally defined as the difference between its total and its internal or rest energy:

$$mc^2 = m_0c^2 + T, \qquad T = m_0c^2(\gamma - 1). \tag{6.10}$$

The leading term in the power-series expansion of T is, of course, the Newtonian $\frac{1}{2}m_0u^2$, as in eqn (6.9); the rest is the relativistic 'correction'. Note that in an *elastic* collision, where by definition each particle's rest-mass is preserved (so that $\sum^* m_0 = 0$), the conservation law $\sum^* m = 0$ implies the conservation of kinetic energy, $\sum^* T = 0$.

Einstein's mass–energy equivalence allows us to include even particles of zero rest-mass (photons, ...) into the scheme of collision mechanics. If such a particle has finite energy E (all of it being kinetic!), it has finite mass $m = E/c^2$ and thus, because of (6.4), it *must* move at the speed of light. Formally we can regard its mass as the limit of a product, γm_0, of which the first factor has gone to infinity and the second to zero. According to (6.5), it then has a perfectly normal 3-momentum $\mathbf{p}$ and thus also a 4-momentum $\mathbf{P}$ given by the last member of eqn (6.3): $\mathbf{P} = (\mathbf{p}, E/c)$. In this case, however, $\mathbf{P}$ is null ($\mathbf{P}^2 = 0$).

In fact, the 4-momentum (6.3) of *any* particle can now be written in the form

$$\mathbf{P} = (\mathbf{p}, E/c). \tag{6.11}$$

Particle physicists, whose basic unit is the electrovolt rather than the gram, prefer this form and tend to discard the concept of relativistic mass altogether. And, of course, they are the main consumers and therefore trend-setters of relativistic mechanics. On the other hand, it is trivial to switch back and forth between m and E and we prefer to keep our options open.

So much for the formalism. What about the applications? For a macroscopic 'particle' the internal energy m_0c^2 is vast: in each gram of mass there are 9×10^{20} ergs of energy, roughly the energy of the Hiroshima bomb (20 kilotons). A very small part of this energy resides in the thermal motions of the molecules constituting the particle, and can be given up as heat; a part resides in the intermolecular and interatomic cohesion forces, and some of that can be given up in chemical explosions; another part may reside in excited atoms and escape in the form of radiation; much more resides in nuclear bonds and can also sometimes be set free, as in the atomic bomb. But by far the largest part of the energy (about 99 per cent) resides simply in the mass of the elementary particles, and cannot be further explained. Nevertheless, it too can be liberated under suitable conditions; for example, when matter and antimatter annihilate each other.

One kind of energy that does *not* contribute to mass is *potential* energy of position. In classical mechanics, a particle moving in an electromagnetic (or gravitational) field is often said to possess potential energy, so that the sum of its kinetic and potential energies remains constant. This is a useful 'book-keeping' device, but energy conservation can also be satisfied by debiting the *field* with an energy loss equal to the kinetic energy gained by the particle. In relativity there are good reasons for adopting the second alternative, though the first can be used as an occasional shortcut: the 'real' location of any part of the energy is no longer a mere convention, since energy (as mass) gravitates; that is, it contributes measurably (in principle) to the curvature of spacetime *at its location*.

According to Einstein's hypothesis, *every* form of energy has a mass equivalent: (i) If all mass exerts and suffers gravity, we would expect even (the energy of) an electromagnetic field to exert a gravitational attraction, and, conversely, light to bend under gravity (this we have already anticipated by a different line of reasoning). (ii) We shall expect a gravitational field *itself* to gravitate. (iii) The radiation which the sun pours into space is equivalent to more than four million tons of *mass* per second! Radiation, having mass and velocity, must also have momentum; accordingly, the radiation from the sun is a (small) contributing factor in the observed deflection of the tails of comets away from the sun. (The major factor is 'solar wind.') (iv) An electric motor (with battery) at one end of a raft, driving a heavy flywheel at the other end by means of a belt, transfers energy and thus mass to the flywheel; in accordance with the law of momentum conservation, the raft must therefore accelerate in the opposite direction. (v) Stretched or compressed objects have (minutely) more mass by virtue of the stored elastic energy. (vi) The total mass of the separate components of a stable atomic nucleus always exceeds the mass of the nucleus itself, since energy (that is, mass) would have to be supplied in order to decompose the nucleus against the nuclear binding forces. This is the reason for the well-known 'mass defect'. Nevertheless, if a nucleus is split into two new nuclei, these parts may have greater *or* lesser mass than the whole. With the lighter atoms, the parts usually exceed the whole, whereas with the heavier atoms the whole can exceed the parts, owing mainly to the electrostatic repulsion of the protons. In the first case, energy can be released by 'fusion', in the second, by 'fission'.

From a historical perspective, Einstein's recognition of $E = mc^2$ did not quite come 'out of the blue'. There had been foreshadowings along those lines for almost a quarter of a century. Already in 1881, J. J. Thomson had calculated that a charged sphere behaves as if it had an *additional* mass of amount $\frac{4}{3}c^{-2}$ times the energy of its Coulomb field. That set off a quest for the 'electromagnetic mass' of the electron—an effort to explain its inertia purely in terms of the field energy. (This effort was beset by the 'wrong' factor $\frac{4}{3}$ due to the as yet unknown mass-equivalent of the stresses needed to hold the 'electron' together.) In 1900, Poincaré made the simpler observation that, since the electromagnetic momentum of radiation is $1/c^2$ times the Poynting flux of energy, radiation seems to possess a mass density $1/c^2$ times its energy density. And then in 1905 came Einstein. What truly sets him apart once more is the universality of his proposal.

6.4 Four-momentum identities

It will be convenient to have a number of often-used identities collected here for future reference. We have already discussed the following alternative expressions for the 4-momentum **P**,

$$\mathbf{P} = m_0\mathbf{U} = (\mathbf{p}, mc) = (\mathbf{p}, E/c), \tag{6.12}$$