Foundations of Physics Letters, Vol. 1, No. 2, 1988

ON THE INERTIAL MASS CONCEPT IN SPECIAL AND GENERAL RELATIVITY

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Received December 15, 1987

Inertial mass in relativity theory is discussed from a conceptual view. It is shown that though relativistic dynamics implies a particular dependence of the momentum of a free particle on its velocity *in special relativity*, which diverges as v approaches c, the inertial mass itself of a moving body remains constant, from any frame of observation. However, extension to general relativity does conceptually introduce variability of the inertial mass of a body, through a necessarily generally covariant field theory of inertia, when the Mach principle is incorporated into the theory of general relativity, as a theory of matter.

Key words: mass, special relativity, general relativity

1. INTRODUCTION

It is often useful in research in theoretical physics, and in the teaching of its concepts, to re-examine the ideas that underlie well accepted formulas, hopefully to gain new understanding of particular aspects of a subject that has been empirically well-represented by these formulas.

In this regard I have re-examined in an earlier paper the meaning of Einstein's energy-mass relation in special relativity[1]

$$E = m_0c^2 \quad \text{(proper frame)} \tag{1}$$
$$m_0c^2/[1 - (v/c)^2]^{\frac{1}{2}} \quad \text{(moving frame)},$$

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demonstrating that, in the frame of the free particle with inertial mass m_0, this equation does not signify that "energy is *equivalent to* mass", as it is usually asserted.

What was pointed out earlier in this regard is that the concept of "energy" per se and the concept of "inertial mass" per se are, firstly, the same as they are in classical physics, and, secondly, that they are logically different concepts: "energy" is defined to be the capacity of matter to do work and "inertial mass" is defined as a quantification of the inertial property of matter, i.e., a measure of its resistance to a change of state of constant speed (or rest) with respect to any observer.

Since these are two entirely different concepts, energy cannot be said to be "equivalent to" mass. It was pointed out further in the earlier paper that in relativity theory, energy and mass also cannot be considered equivalent, because there are domains where energy is not even defined, as in the global extension to general relativity, where there are no conservation laws, yet where the concept of inertial mass remains. Thus, rather than saying that mass is equivalent to energy, one should say that once one has in hand the (expectation value of the) inertial mass of matter in the local domain, then one may determine its potential to do work, i.e., its intrinsic energy, by using the formula from special relativity, $E = m_0c^2$.

Indeed, in the paper in which Einstein derived this relation[2], he did not claim that energy is equivalent to mass. What he did conclude from the appearance of this formula was: "The mass of a body is a measure of its energy content". Clearly, a *measure* of the body's energy content does not refer to the equivalence of mass and energy - just as the speed of a body in classical physics, being a measure of its kinetic energy, does not signify that speed is equivalent to kinetic energy!

2. MOMENTUM IN SPECIAL RELATIVITY

In this note I wish to extend the discussion of special relativity as a limiting case of general relativity, regarding dynamical concepts, to the usually accepted interpretation that inertial mass is a velocity-dependent parameter. I do not believe that this is conceptually true within the structure of relativity theory, though the formula it is derived from is true. This formula is the relation between the momentum of a free particle in special relativity and its velocity, from the frame of an observer who "sees" the particle in motion at the speed v:

$$\vec{p} = m_o\vec{v}/[1 - (v/c)^2]^{\frac{1}{2}} , \tag{2}$$

where m_o is the rest mass of the particle and c is the speed of light.

This formula is, of course, empirically valid and substantiates predictions of the theory of special relativity. However, it does not specify the variability of the mass. In the early stages of special relativity theory it was claimed by several authors(3) that the relationship (2) implies that there is a speed-dependence of the inertial mass as follows:

$$m(v) = m_o/[1 - (v/c)^2]^{\frac{1}{2}} \tag{3}$$

whereby the expression for the momentum of a freely moving particle would then take the form:

$$\vec{p}(v) = m(v)\vec{v} \tag{3'}$$

However, looking at the roots of this formula, we see that the expression for m(v) in Eq. (3) is *not in itself* the inertial mass of the body. Rather, it is a combination of factors, as in Eq. (3), which restores the classical expression (3') for the momentum. Thus, one may not say that, because of Eq. (3), the "inertial mass" of a body in itself increases toward infinity as v approaches c.

What is the meaning of the inertial mass of a body in the theory of special relativity? As it was pointed out earlier, it is the same concept as we encounter in classical Newtonian physics - a measure of the body's resistance to a change of state of constant speed (or rest). This feature of matter is quantified with the *constant* parameter m_o.

The expressions (1) and (2) for the energy and momentum in the dynamics of special relativity are found to originate, mathematically, in the stationarity of the path integral (i.e., its vanishing variation) over the invariant differential of special relativity theory:

$$\delta\int ds = 0 , \tag{4}$$

where the differential invariant is, in squared form,

$$ds^2 = c^2dt^2 - dr^2 .$$

This vanishing variation then yields the geodesic path. Thus the action functional for the free particle is taken to be proportional to this invariant, i.e.,[4]

$$S = a\int ds \tag{5}$$

where a is the proportionality constant, to be determined.

In the proper frame of reference of the free particle, with the differential interval of special relativity specified above, this action functional may then be written as

$$S = ac\int[1 - (v/c)^2]^{\frac{1}{2}}dt \tag{6}$$

In the asymptotic limit of classical physics (in accordance with the correspondence principle), where $v/c << 1$, one may then make the identification with the classical Lagrangian

$$L_{cl} = ac[1 - (v/c)^2]^{\frac{1}{2}}$$
$$\simeq ac - \tfrac{1}{2}av^2/c \qquad (v/c << 1).$$

Classically, Lagrange's equation of motion then gives for the total energy, which in this case is the kinetic energy,

$$E = \tfrac{1}{2}m_o v^2 = -\tfrac{1}{2}av^2/c ,$$

where m_o is the constant inertial mass of the particle. Thus, one has the identification $a = -m_o c$ for the proportionality constant in (6), so that the action functional in (6) becomes

$$S = -m_o c^2\int[1 - (v/c)^2]^{\frac{1}{2}}dt .$$

In the Lagrangian formalism, the vanishing variation of S with respect to time, $\delta S/\delta t = 0$, yields the energy relation (1), while the vanishing variations with respect to the three spatial coordinates, $\delta S/\delta x_i = 0$, yield the momentum relations (2) of special relativity theory.

The preceding derivations of the energy and momentum of a free particle in relativistic dynamics are well known. What I wish to point out here is that the origin of the inertial mass parameter m_o, per se, in special relativity follows from these derivations.

What is clear is that a particle with mass m_o, approaching the speed of light c does not affect the actual inertial mass of that body, which is its rest mass, though the components of its momentum do correspondingly approach infinity, from an observer's view, in accordance with Eq. (2). Of course, this result of special relativity has had spectacular observable consequences. For example, if a nuclear particle, traveling close to the speed of light, is stopped rapidly in a nucleus, in Δt sec, then the average force it imparts to the nucleus during this time is $\bar{F} = (p - 0)/\Delta t = p/\Delta t$. With the relativistic momentum (2), with v close to c and very small Δt, this force is much greater than would be expected on classical grounds - of order $[1 - (v/c)^2]^{-\frac{1}{2}}$ where v/c is close to unity. All the while, however, the actual quantity of inertial mass of the projectile particle is its rest mass, m_o (not m(v)). That is to say, in special relativistic dynamics the actual inertial mass of the body does not change in the process of momentum transfer. Its magnitude is an intrinsic feature of the particle that is considered. Its constancy expresses the restraint on the variability of the particle's energy and momentum, in accordance with Eqs. (1) and (2), as follows:

$$m_o = (E^2 - p^2c^2)^{\frac{1}{2}}/c^2$$

3. INERTIAL MASS IN GENERAL RELATIVITY

The conclusion we have reached so far is that it is not meaningful *in special relativity* to refer to a velocity-dependent mass of a free particle. In the context of special relativity the only inertial mass is the rest mass, m_o.

On the other hand, if a particle of matter is not free, special relativity does not in principle apply. One must then globally extend the representations of the laws of nature in a Euclidean (flat) space-time to the (curved) Riemannian space-time. This is because the description of the "particle", from the frame of reference of the exerting force, is not inertial. One must then in principle extend the formalism to general relativity.

If one takes the interpretation of inertial mass according to the Mach principle - implying that, rather than being intrinsic, mass is a measure of dynamical coupling with the rest of a closed system, then it follows that the most primitive expression for inertial mass must be in terms of a continuously variable *matter field*, rather than a constant parameter. In general relativity, this field then must solve a generally covariant field equation, coupled to the force-field equations of the theory. (It is the latter field equation that I have found yields a formal expression of quantum mechanics, in a low energy, linear approximation.[5]

Thus, taking general relativity as a fundamental theory of matter, in accordance with Einstein's original view, that extends his formalism so as to incorporate the Mach principle in his field theory, we see that inertial mass is indeed continuously variable, depending on all of the other variables that define the environment of an interacting component of a system - whose inertial mass we are considering. It is only in the low energy, linear limit, where the generally covariant mass field equations approach the formal expression of quantum mechanics in special relativity (and then to the nonrelativistic approximations of the Schrödinger or Heisenberg forms), that one may replace this inertial mass field with its average value, locally; that is, m_o, assuming the latter to be the intrinsically constant inertial mass of the particle of matter.

Even though this can be an accurate approximation when there is nonrelativistic energy-momentum transfer, it should be noted that in the exact limit, when special relativity (and Euclidean space-time) is used, this can only represent, in its exact form, the free particle in a vacuum (everywhere), since only then would there be no force exerted on the particle, so that it would truly be in an inertial reference frame with respect to any observer. This (ideal) limit must then correspond exactly to a zero value for the inertial mass of the particle - since it would no longer be coupled to any other matter.

This conclusion is, of course, in accordance with the requirement of the Mach principle. It is a prediction that cannot be tested directly - since one can never, in principle, form a perfect vacuum. But it can be tested indirectly, looking for a *change* in the magnitude of the inertial mass as a function of large changes in the environment of an "observed" particle of matter. Such observations should be testable in the very-high-energy physics experimentation of the near future.

One further point is the following: If the inertial mass of a particle of matter is a consequence of its dynamical coupling with its material environment, then it should also follow that so must all other seemingly "intrinsic properties" of this particle be consequences of the dynamical coupling to the environment of the observed particle. I have referred to this idea as the "Generalized Mach Principle", and I have demonstrated it explicitly in regard to the electromagnetic properties of a "seeming" particle of matter, in general relativity theory.[6] It is a view that removes all remnants of atomism from the field representation of matter, in accordance with the conceptual view of Einstein's theory of general relativity.

ACKNOWLEDGEMENTS

I thank the staff of the Edelstein Center for the History and Philosophy of Science, Medicine and Technology, of the Hebrew University of Jerusalem, and its Director, Dr. Itamar Pitowsky, for kind hospitality and for granting me a Senior Visiting Fellowship during my 1986-87 sabbatical year, when this paper was prepared.

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