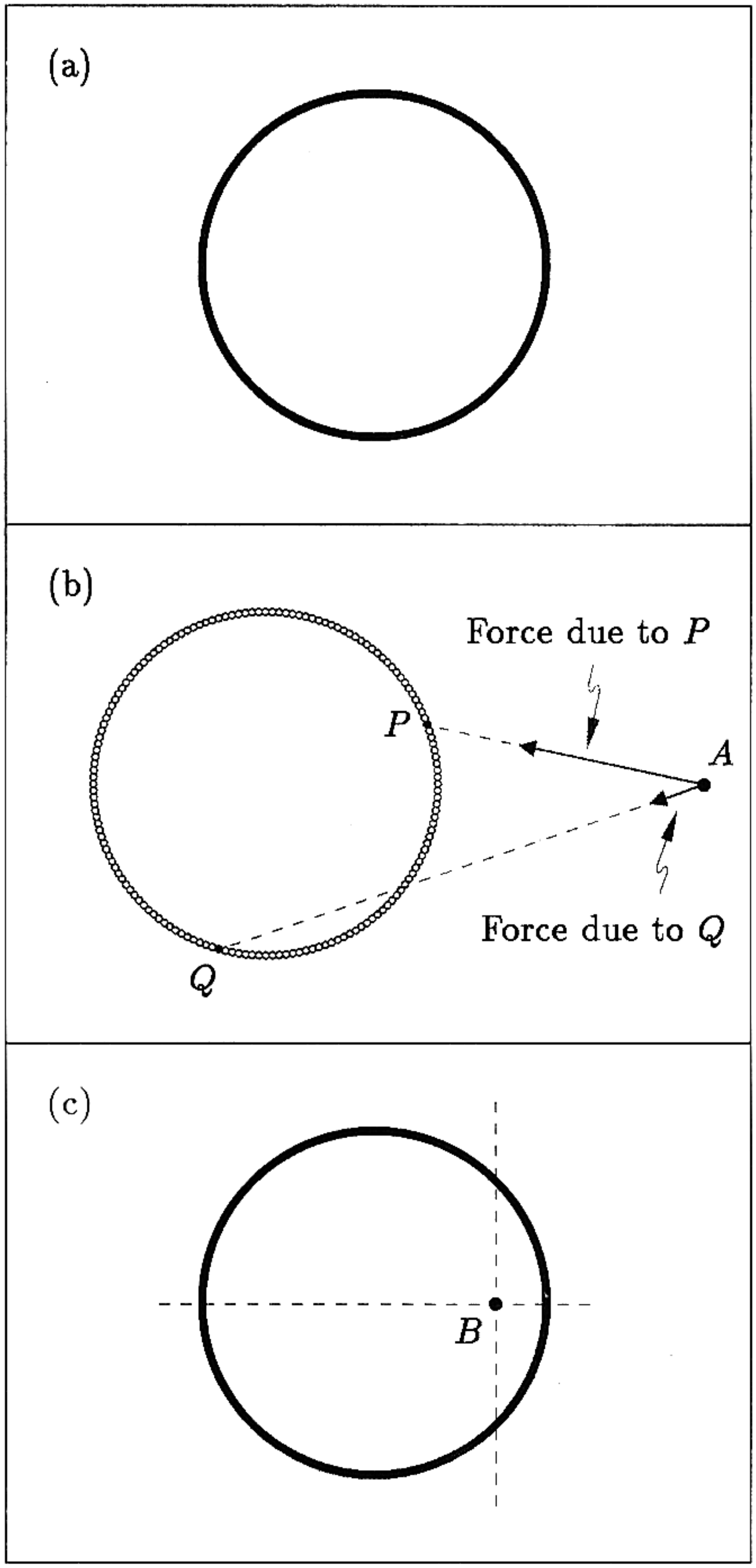
APPENDIX A

GRAVITATIONAL ENERGY

Since the negative energy of a gravitational field is crucial to the notion of a zero-energy universe, it is a subject worth examining carefully. In this appendix I will explain how the properties of gravity can be used to show that the energy of a gravitational field is unambiguously negative. The argument will be described in the context of Newton's theory of gravity, although the same conclusion can be reached using Einstein's theory of general relativity.

A simple way to demonstrate the sign of gravitational energy is to imagine a thin spherical shell of mass, as shown in Figure A.1(a). The shell will create a gravitational field, which at each point in space provides a measure of the force that would be experienced by a mass if it were located at that point. The gravitational field can be calculated by using Newton's methods. Newton first considers an ideal point mass—a mass concentration that is so small that it can be treated as if all the mass were located at a single point in space. For this case the gravitational field points directly toward the mass, with a strength that is described by the inverse square law—that is, the strength decreases as the square of the distance from the point mass. For a more complicated object such as the shell of Figure A.1(a), the gravitational field is in principle determined by mentally dividing the object into an infinite number of point masses, each with a mass that is infinitely small. A schematic illustration of this division into point masses is shown in part (b) of the figure. For each point mass one uses the simple inverse square law, and then one has to add the infinite number of contributions to obtain the final answer. The technique for handling such



Thin
Spherical
Shell

Calculation
of Force
on Point
Outside Shell

Calculation
of Force
on Point
Inside Shell

an infinite number of quantities is the main subject of what is called integral calculus, which was largely developed by Newton himself.

The result for the gravitational field of a spherical shell of mass was first calculated by Newton, and it is the sort of calculation that is likely to show up in any college-level course in physics. Newton found that, outside the shell, the gravitational field at any point is directed radially inward toward the center of the shell. This answer could have been anticipated from the symmetry of the problem: there is no other reasonable answer to the question "What direction could it possibly point?" The strength of the gravitational field outside the shell can be described with surprising simplicity: the gravitational field has exactly the same strength as it would if all of the mass were concentrated at the center point of the shell.

What is the gravitational field inside the spherical shell? Consider, for example, the force on a particle at position *B*, as shown in part (c). By symmetry the force will be along the horizontal broken line, because the upward force caused by attraction toward the mass in the upper half of the diagram will be canceled by an opposing downward attraction toward the mass in the lower half of the diagram. We still must decide, however, if the force will point to the right or to the left. There is a persuasive argument which says that the force should be to the right: since the matter to the right is much closer than the matter to the left, the inverse square law should mean that the attraction to the matter on the right should dominate. There is also, however, a persuasive argument which says that the force should be to the left: There is more matter on the left, so the attraction toward it should dominate.

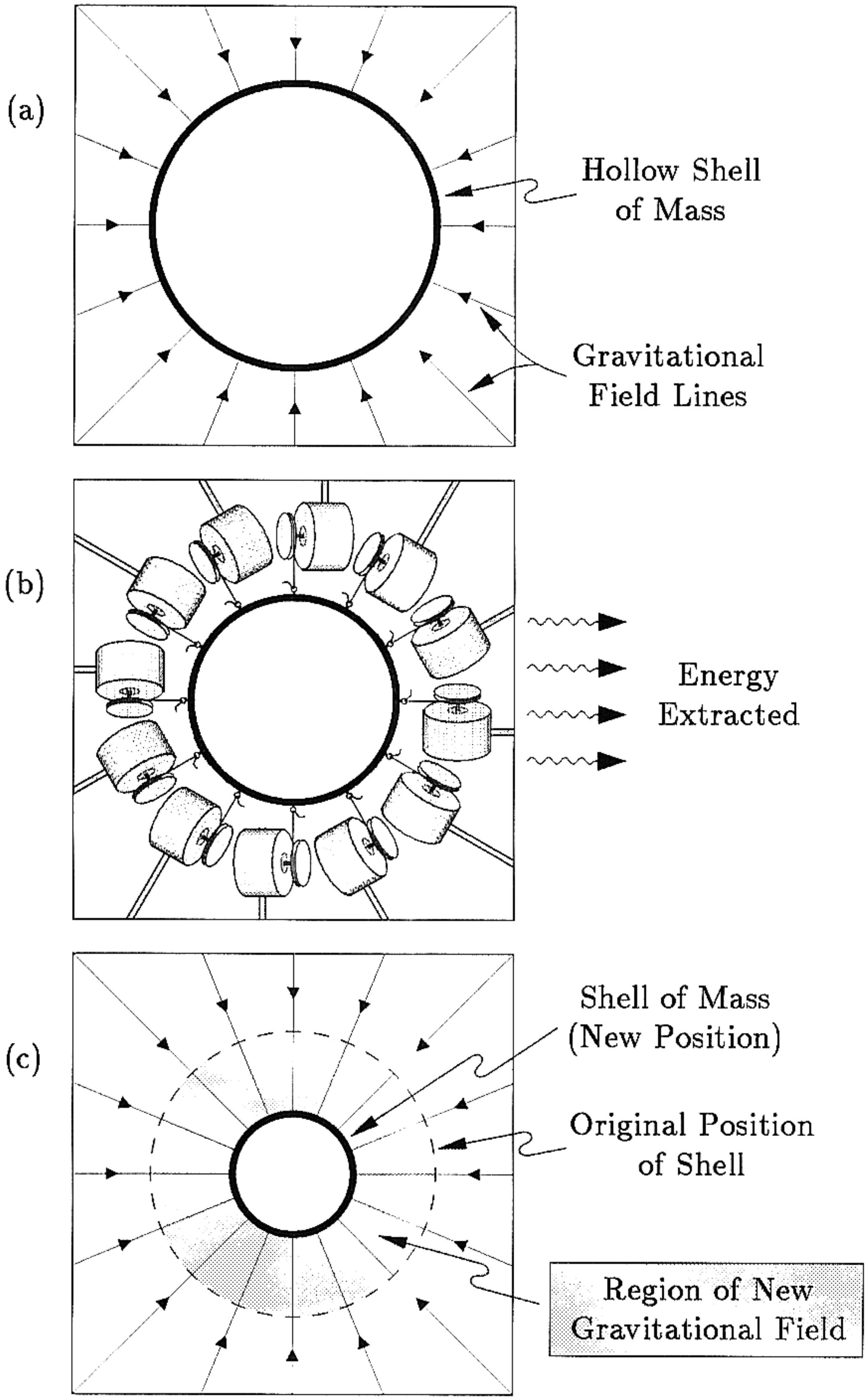
Which of the arguments in the previous paragraph is correct? Newton showed that these two arguments are equally valid, and in fact the forces cancel out exactly for a particle placed at any point inside the spherical cavity.

***Figure A.1* (facing page) The gravitational field of a thin spherical shell.** Part (a) shows the thin shell. The shell is really a three-dimensional sphere, but the diagram shows only a two-dimensional slice through the center of the sphere. In part (b) the shell has been divided into a large number of point masses, for the purpose of calculating the gravitational field at the point *A*. In principle there should be an infinite number of mass points, but only a finite number can be shown. The arrows indicate the gravitational attraction toward the mass points labeled *P* and *Q*. The attraction toward *P* is larger, because it is closer. Part (c) shows a point *B*, inside the spherical cavity, at which the gravitational field is to be calculated. The mass to the right of the vertical broken line is closer than the mass to the left, but there is more mass to the left.

To proceed with the discussion of gravitational energy, we need to answer one more question: How will gravity affect the spherical shell itself? Each point on the spherical shell will be attracted gravitationally toward each of the other points on the shell, and the net effect is a force pulling each point toward the center of the sphere. If the matter from which the shell is constructed is soft and compressible, then gravity will cause the shell to contract.

The situation is illustrated in Figure A.2, where part (a) shows the thin shell and the gravitational field that it generates. Outside the shell the gravitational field points inward, and inside the gravitational field is zero. Now imagine what would happen if the shell were allowed to uniformly contract, keeping its spherical shape. One can imagine, for example, extracting energy by tying ropes to each piece of the shell, as is illustrated in part (b). These ropes can be used to drive electric generators as each piece is lowered to its new position. Part (c) shows the sphere after the new radius is attained. The dashed circle indicates the original radius of the shell, and outside of the dashed circle the gravitational field is identical to that in part (a). (Recall that the field outside is the same as if all the mass were concentrated at the center, so it does not depend on the radius of the shell.) Inside the shell in its new position, the gravitational field remains zero. However, in the shaded region between the original and new positions of the shell, a gravitational field now exists where no field had existed before. The net effect of this operation is to *extract* energy, and to create a *new* region of gravitational field. Thus, energy is *released* when a gravitational field is created. The energy contained in the shaded region must therefore decrease, just as the water level in a tank decreases if water is released. Since the region began with no gravitational field and hence no energy, the final energy must be *negative*. In most physical processes the exchange of gravitational energy is much smaller than the rest energy (mc^2) of the particles involved, but cosmologically the total gravitational energy can be very significant.

Figure A.2 **(facing page) Thought experiment to understand the energy of gravity.** Part (a) shows a hollow spherical shell of mass, and the gravitational field lines that it produces. There is a force on each piece of the shell, pulling inward. Part (b) shows how energy can be extracted as the shell is allowed to uniformly contract. Each piece of the shell is tied by a rope to an electrical generator, producing power as the piece is "lowered" toward its final position. Part (c) shows the final configuration, which includes a gravitational field in the shaded region where no field existed before. Thus, the creation of the gravitational field is associated with the release of energy.


(a)
Hollow Shell of Mass
Gravitational Field Lines
(b)
Energy Extracted
(c)
Shell of Mass (New Position)
Original Position of Shell
Region of New Gravitational Field